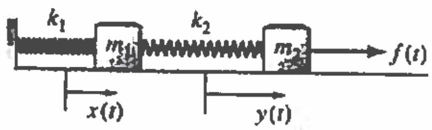
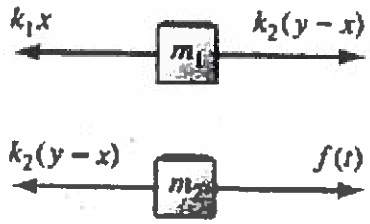
Solutions

4.1: First Order Systems and Applications

Example 1.



Equilibrium positions



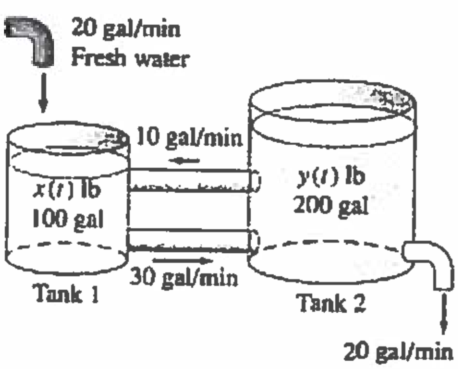
Consider the system of two masses and two springs shown above left, with an external force $f(t)$ acting on the right mass m_2. Applying Newton's law of motion to the two "free body diagrams" shown top right, we obtain the system

$$\begin{aligned} m_1 x'' &= -k_1 x + k_2(y-x) \\ m_2 y'' &= -k_2(y-x) + f(t). \end{aligned}$$

If, for instance, $m_1 = 2, m_2 = 1, k_1 = 4$ and $f(t) = 40\sin 3t$ then we arrive at

$$\begin{aligned} 2x'' &= -6x + 2y \\ y'' &= 2x - 2y + 40\sin 3t. \end{aligned}$$

Example 2.



Consider two brine tanks (shown left). Tank 1 contains $x(t)$ pounds of salt in 100 gal of brine and tank 2 contains $y(t)$ pounds of salt in 200 gal of brine. Everything is kept uniform by stirring as tank 1 receives 20 gal/min of fresh water and tank 2 flows out at 20 gal/min. Computing the rate of change of salt in each tank, we arrive at

$$\begin{aligned} x' &= -30 \cdot \frac{x}{100} + 10 \cdot \frac{y}{200} \\ y' &= 30 \cdot \frac{x}{100} - 10 \cdot \frac{y}{200} - 20 \cdot \frac{y}{200}. \end{aligned}$$

In this section we wish to only consider first-order systems. In order to do this, we will change higher-order systems into first-order systems.

Example 3. Rewrite the third-order system

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

as an equivalent first-order system of equations.

$$x^{(3)} = f(t, x, x', x'') = 5x - 2x' - 3x'' + \sin 2t$$

Let $x_1 = x, \quad x_2 = x', \quad x_3 = x''.$

Then

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 5x_1 - 2x_2 - 3x_3 + \sin 2t$$

Example 4. Rewrite the second-order system

$$2x'' = -6x + 2y$$
$$y'' = 2x - 2y + 40 \sin 3t$$

as an equivalent first-order system of equations.

Let $x_1 = x, \; x_2 = x', \; y_1 = y, \; y_2 = y'.$

Then

$$x_1' = x_2$$
$$2x_2' = -6x_1 + 2y_1$$
$$y_1' = y_2$$
$$y_2' = 2x_1 - 2y_1 + 40 \sin 3t.$$

Example 5. Solve the two-dimensional system

$$x' = -2y, \quad y' = \frac{1}{2}x.$$

$$x'' = (-2y)' = -2y' = \cancel{} -x.$$

Or $$x'' + x = 0$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$x = A\cos t + B\sin t$$

So $$y = -A\sin t + B\cos t$$

Using trig, we can see $\dfrac{x^2}{C^2} + \dfrac{y^2}{(C/2)^2} = 1$ $\left(C = \sqrt{A^2+B^2}\right)$

Example 6. Find the general solution of the system

$$x' = y, \quad y' = 2x + y.$$

$$x'' = y' = 2x + y = x' + 2x$$

or

$$x'' - x' - 2x = 0$$

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0$$

So $x = Ae^{-t} + Be^{2t}$.

and $y = -Ae^{-t} + 2Be^{2t}$.

Example 7. Solve the initial value problem

$$x' = -y,$$
$$y' = (1.01)x - (0.2)y,$$
$$x(0) = 0, \quad y(0) = 1.$$

$$x'' = -y' = (0.2)y - (1.01)x = -(0.2)x' - (1.01)x$$

or

$$x'' + (0.2)x' + (1.01)x = 0$$
$$r^2 + (0.2)r + 1.01 = 0$$
$$(r + 0.1)^2 + 1 = 0$$
$$r = -0.1 \pm i$$

So $x = e^{-t/10}(A\cos t + B\sin t) \Rightarrow A = 0$

and $y = \frac{1}{10} e^{-t/10} \cdot B\sin t - Be^{-t/10}\cos t \Rightarrow B = -1$

So $x = -e^{-t/10}\sin t$

$y = \frac{1}{10} e^{-t/10}(10\cos t - \sin t)$

Theorem 1. (Existence and Uniqueness for Linear Systems)
Suppose that the functions $p_{11}, p_{12}, \ldots, p_{nn}$ and the functions $f_1, \ldots, f_n$ are continuous on the open interval I containing a. Then, given the n numbers $b_1, \ldots, b_n$, the linear system has a unique solution on the entire interval I that satisfies the n initial conditions

$$x_1(a) = b_1, \quad \cdots, \quad x_n(a) = b_n.$$

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t)$$
$$\vdots$$
$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).$$

Homework. 1-7, 17-25 (odd)